Problem 1.17

Suppose you wanted to describe an **unstable particle**, that spontaneously disintegrates with a "lifetime" τ. In that case the total probability of finding the particle somewhere should *not* be constant, but should decrease at (say) an exponential rate:

$$P(t) \equiv \int_{-\infty}^{+\infty} |\Psi(x,t)|^2 \, dx = e^{-t/\tau}.$$

A crude way of achieving this result is as follows. In Equation 1.24 we tacitly assumed that V (the potential energy) is *real.* That is certainly reasonable, but it leads to the "conservation of probability" enshrined in Equation 1.27. What if we assign to V an imaginary part:

$$V = V_0 - i\Gamma,$$

where V_0 is the true potential energy and Γ is a positive real constant?

(a) Show that (in place of Equation 1.27) we now get

$$\frac{dP}{dt} = -\frac{2\Gamma}{\hbar} P.$$

(b) Solve for $P(t)$, and find the lifetime of the particle in terms of Γ.

Solution

Note that the governing equation for the wave function is Schrödinger's equation.

$$\begin{aligned}
\frac{\partial \Psi}{\partial t} &= \frac{i\hbar}{2m} \frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar} V(x,t)\Psi(x,t) \\
&= \frac{i\hbar}{2m} \frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar} [V_0(x,t) - i\Gamma]\Psi(x,t) \\
&= \frac{i\hbar}{2m} \frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar} V_0(x,t)\Psi(x,t) + \frac{i^2\Gamma}{\hbar}\Psi(x,t) \\
&= \frac{i\hbar}{2m} \frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar} V_0(x,t)\Psi(x,t) - \frac{\Gamma}{\hbar}\Psi(x,t)
\end{aligned}$$

Take the complex conjugate of both sides to get the corresponding equation for Ψ^*.

$$\begin{aligned}
\frac{\partial \Psi^*}{\partial t} &= -\frac{i\hbar}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} V^*(x,t)\Psi^*(x,t) \\
&= -\frac{i\hbar}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} [V_0(x,t) + i\Gamma]\Psi^*(x,t) \\
&= -\frac{i\hbar}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} V_0(x,t)\Psi^*(x,t) + \frac{i^2\Gamma}{\hbar}\Psi^*(x,t) \\
&= -\frac{i\hbar}{2m} \frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar} V_0(x,t)\Psi^*(x,t) - \frac{\Gamma}{\hbar}\Psi^*(x,t)
\end{aligned}$$

Now consider the derivative of $P(t)$ with respect to t.

$$\begin{aligned}
\frac{dP}{dt} &= \frac{d}{dt}\int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx \\
&= \frac{d}{dt}\int_{-\infty}^{\infty} \Psi(x,t)\Psi^*(x,t)\,dx \\
&= \int_{-\infty}^{\infty} \frac{\partial}{\partial t}[\Psi(x,t)\Psi^*(x,t)]\,dx \\
&= \int_{-\infty}^{\infty} \left(\frac{\partial\Psi}{\partial t}\Psi^* + \Psi\frac{\partial\Psi^*}{\partial t}\right)dx \\
&= \int_{-\infty}^{\infty} \left[\left(\frac{i\hbar}{2m}\frac{\partial^2\Psi}{\partial x^2} - \frac{i}{\hbar}V_0\Psi - \frac{\Gamma}{\hbar}\Psi\right)\Psi^* + \Psi\left(-\frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2} + \frac{i}{\hbar}V_0\Psi^* - \frac{\Gamma}{\hbar}\Psi^*\right)\right]dx \\
&= \int_{-\infty}^{\infty} \left(\frac{i\hbar}{2m}\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \cancel{\frac{i}{\hbar}V_0\Psi^*\Psi} - \frac{\Gamma}{\hbar}\Psi^*\Psi - \frac{i\hbar}{2m}\frac{\partial^2\Psi^*}{\partial x^2}\Psi + \cancel{\frac{i}{\hbar}V_0\Psi^*\Psi} - \frac{\Gamma}{\hbar}\Psi^*\Psi\right)dx \\
&= \int_{-\infty}^{\infty} \left[\frac{i\hbar}{2m}\left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi^*}{\partial x^2}\Psi\right) - \frac{2\Gamma}{\hbar}\Psi^*\Psi\right]dx \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty} \left(\Psi^*\frac{\partial^2\Psi}{\partial x^2} - \frac{\partial^2\Psi^*}{\partial x^2}\Psi\right)dx - \frac{2\Gamma}{\hbar}\int_{-\infty}^{\infty} \Psi^*(x,t)\Psi(x,t)\,dx \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty} \left[\left(\frac{\partial\Psi^*}{\partial x}\frac{\partial\Psi}{\partial x} + \Psi^*\frac{\partial^2\Psi}{\partial x^2}\right) - \left(\frac{\partial^2\Psi^*}{\partial x^2}\Psi + \frac{\partial\Psi^*}{\partial x}\frac{\partial\Psi}{\partial x}\right)\right]dx - \frac{2\Gamma}{\hbar}\int_{-\infty}^{\infty} |\Psi(x,t)|^2\,dx \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty} \left[\frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\Psi^*}{\partial x}\Psi\right)\right]dx - \frac{2\Gamma}{\hbar}P(t) \\
&= \frac{i\hbar}{2m}\int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)dx - \frac{2\Gamma}{\hbar}P(t) \\
&= \frac{i\hbar}{2m}\underbrace{\left(\Psi^*\frac{\partial\Psi}{\partial x} - \frac{\partial\Psi^*}{\partial x}\Psi\right)\Bigg|_{-\infty}^{\infty}}_{=0} - \frac{2\Gamma}{\hbar}P(t) \\
&= -\frac{2\Gamma}{\hbar}P(t)
\end{aligned}$$

Divide both sides by $P(t)$.

$$\frac{\frac{dP}{dt}}{P(t)} = -\frac{2\Gamma}{\hbar}$$

The left side can be written as the derivative of a logarithm by the chain rule.

$$\frac{d}{dt}\ln P(t) = -\frac{2\Gamma}{\hbar}$$

Integrate both sides with respect to t.

$$\ln P(t) = -\frac{2\Gamma}{\hbar}t + C$$

Exponentiate both sides.

$$\begin{aligned}
P(t) &= e^{-2\Gamma t/\hbar + C} \\
&= e^{-2\Gamma t/\hbar}e^C
\end{aligned}$$

Use a new constant A for e^C.

$$P(t) = Ae^{-2\Gamma t/\hbar}$$

Assume that the particle is somewhere initially so that $P(0) = 1$. Then A can be determined.

$$P(0) = A = 1$$

Therefore,

$$P(t) = e^{-2\Gamma t/\hbar} = e^{-t/\tau},$$

where the lifetime is

$$\tau = \frac{\hbar}{2\Gamma}.$$